Analysis on Manifolds (WBMA013-05)– Resit

Friday 9 April 2021, 8:30h–11:30h

This exam consists of **3** problems.
Usage of the theory and examples from the lecture notes is allowed. Give a precise reference to the theory and/or exercises you use for solving the problems.
You get 10 points for free.

Problem 1. (8 + 8 + 7 + 7 = 30 points)

Let $M = \mathbb{R}^3$, $\eta := dz - \frac{1}{2}(xdy - ydx) \in \Omega^1(M)$ and $X, Y, Z \in \mathfrak{X}(\mathbb{R}^3)$ defined by $X := \frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial z}$, $Y := \frac{\partial}{\partial y} + \frac{x}{2}\frac{\partial}{\partial z}$ and $Z := \frac{\partial}{\partial z}$.

(a) Show that $\mathscr{D} = \ker(\eta)$ is a linear subspace of $T\mathbb{R}^3$;
Hint: if you are not sure what to do, you can proceed as follows. For $q \in M$, denote $\mathscr{D}_q := \ker(\eta_q) \subset T_qM$. For all $q \in M$, show that $\mathscr{D}_q$ is closed under addition and multiplication, and contains 0_q.

(b) Show that $\{X, Y\}$ is a basis for $\mathscr{D}$;

(c) Compute $[X, Y]$.

(d) Is the vector field $[X, Y]$ an element of $\mathscr{D}$? Justify your answer.

Problem 2. (7 + 8 + 8 + 7 = 30 points)

In this exercise we keep using the same notation of Problem 1. Let $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ be the projection on the (x, y)-plane, that is, $\pi(x, y, z) = (x, y)$.

(a) Show that $\eta \wedge d\eta$ defines a volume form on M.

Let $\gamma = (\gamma^1, \gamma^2, \gamma^3) : I \subset \mathbb{R} \to \mathbb{R}^3$ be a smooth curve in M such that the following holds: (1) for all $t \in I$, $\dot{\gamma}(t) \in \mathscr{D}_{\gamma(t)}$; (2) $\pi \circ \gamma = (\gamma^1, \gamma^2)$ is a closed (possibly self-intersecting) planar curve.

(b) Show that $\int_\gamma \frac{1}{2}(xdy - ydx)$ coincides with the signed area of the region $\Omega \subset \mathbb{R}^2$ bounded by the curve $\pi \circ \gamma$;
Hint: what is the planar integral to compute the area of Ω?

(c) Show that condition (1) in the definition of γ implies that $\gamma^3(t)$ is proportional to the signed area of the region Ω bounded by $\pi \circ \gamma$ on the (x, y)-plane;
Hint: compute $\eta_{\gamma(t)}(\dot{\gamma}(t)) = 0$ and integrate the resulting expression

(d) Show that γ is a closed curve in M if and only the signed area of Ω vanishes.

Problem 3. (10 + 10 + 10 = 30 points)

Let now M be a smooth manifold of dimension $2n$ equipped with a non-degenerate closed form $\omega \in \Omega^2(M)$. Note that if V is a vector space, we say that $\omega : V \times V \to \mathbb{R}$ is *non-degenerate* if $\omega(u, v) = 0$ for all $v \in V$ implies $u = 0$.

(a) Show that for any $H \in C^\infty(M)$ there exists a unique vector field $X_{H,\omega}$ such that

$$\iota_{X_{H,\omega}} \omega = dH. \tag{1}$$

Hint: show that $X \mapsto \iota_X \omega : TM \to T^*M$ *defines an isomorphism.*

(b) Assume that $X_{H,\omega}$ is complete and let $\phi_t := \phi_t^{X_{H,\omega}}$ denote its flow. Prove that the H is constant along the flow of X_H, that is $\frac{d}{dt}(\phi_t^* H) = 0$.

Let now $M = T^*N$ be the cotangent bundle of a smooth n-manifold N. Any chart (U, ϕ) of M induces a chart $(T^*U, \widetilde{\phi})$ on T^*M whose local coordinates are of the form $(x^1, \dots, x^n, \xi_1, \dots, \xi_n)$, where (x^i) denote the local coordinates on $U \subset M$ and (ξ_i) the components of the local expression of covectors $\xi_i dx^i$.

(c) Assume that with such local coordinates on U, $\omega = d\xi_i \wedge dx^i$. Compute X_H in local coordinates.
Hint: write $X_H = A^i \frac{\partial}{\partial x^i} + B_i \frac{\partial}{\partial \xi_i}$.